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THE THEORY OF PLASTICITY IN THE CASE OF SIMPLE LOADING ACCOMPANIED BY STRAIN-HARDENING

By

A. A. Ilyushin

Translation of "Teoriya Plastichnosti pri Prostom Naruzhenii Tyel, Material kotorikh Obladayet Uprochneniem" from Prikladnaya Matematika i Mekhanika XI, No. 2, 1947

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The states of stress and strain of bodies are characterized by the stress tensor S and the strain tensor E, each of which is usually represented in the form of the sum of a spherical tensor and the deviator

$$\left.\begin{aligned} S &= \sigma I + D_s \\ E &= eI + D_e \end{aligned}\right\} \tag{1}$$

so that I is a unit tensor, σ and e the mean values of the diagonal components of S and E or of their linear invariants. The deviators in turn are represented in the form

$$\left.\begin{aligned} D_s &= \tau_i D_s{}^* \\ D_e &= \frac{\gamma_i}{2} D_e{}^* \end{aligned}\right\} \tag{2}$$

where τ_i and γ_i are the quadratic invariants of the deviators D_s and D_e, known as the octahedral shear stress (stress intensity) and the octahedral shear strain (strain intensity):

$$\left.\begin{aligned} \tau_i &= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\ \gamma_i &= \frac{2}{3}\sqrt{(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2} \end{aligned}\right\} \tag{3}$$

Hence σ_n and e_n are the principal components of the S and E tensors.

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*"Teoriya Plastichnosti pri Prostom Naruzhenii Tyel, Material kotorikh Obladayet Uprochneniem." Prikladnaya Matematika i Mekhanika XI, No. 2, 1947, pp. 293–296.

The deviators D_s^* and D_e^* may be called directed tensors, and corresponding to them are Cauchy surfaces, directed hyperboloids. Each of these is completely determined by four numbers, of which three fix the orientation of the principal axes 1, 2, 3 relating to the arbitrary system of coordinates (for example, through the Euler angles), and the fourth determines the ratio between any pair of components in the deviators or the ratio between any pair of semiaxes of the Cauchy surfaces.

In fact, between the six components of the deviator $D_e\{S_{mn}\}$ there exist two relations

$$\left.\begin{aligned} S_{11}^* + S_{22}^* + S_{33}^* &= 0 \\ (S_{11}^* - S_{22}^*)^2 + (S_{22}^* - S_{33}^*)^2 + (S_{33}^* - S_{11}^*)^2 & \\ + 6\left(S_{12}^{*2} + S_{23}^{*2} + S_{31}^{*2}\right) &= 9 \end{aligned}\right\} \qquad (4)$$

and so only four of them are independent. The same holds for the components of the deviator $D_e\{E_{mn}\}$.

The state of stress and strain of an element of a body depends on one parameter λ. Such a parameter may be, for example, the time or a value of the load. If the invariant τ_i increases with increasing λ, one may say that the element is being loaded; if it decreases with increasing λ, the element is being unloaded.

Simple loading is defined as that for which the directed stress tensor D_s^* does not change with increasing λ; the directed stress hyperboloid in this case remains fixed for each element. Other cases of loading are called complex.

The introduction of the above concept, its qualification, and the principal difficulties so far unresolved are susceptible to easy realization in the present state of plasticity theory. Through analysis of the experimental data connected with establishment of the laws of plasticity, we arrive at the following conclusions:

1. The laws of plasticity are completely established only for the case of simple loading of an element of the body; indeed, the tests of Ros and Eichinger, Schmidt, Lode and Nadai, Taylor, Davis and Nadai and others (reference 1) show that tube experiments under complex stress conditions always yield a single result if the relative elements of the principal stress axes are unchanged, and if the ratio of the two principal stresses is constant during the tests; this result leads to the conclusion that an invariant dependency exists between stress and strain.

2. In the case of complex loading of an element, on the contrary, no determinate relationship has been established: the Bauschinger effect, the mutual orientation of the principal axes of stress and strain, the strain velocity, the strain-hardening effect and other phenomena in the complex stress state are not studied. It is clear only that the laws holding for simple loading do not hold for complex loading.

One asks, is not the theory of plasticity, known to be correct only for simple loading, excessively restricted and is it sufficiently broad for the solution of practical problems? In answer to this one may say the following: firstly, such a theory and within such limits of applicability is unique, in agreement with experiment, and from this it follows, secondly, it is applied and gives correct (agreeing with tests) results for a large class of important technical problems.

The following assertion flows from the theorem which we have demonstrated in previous work (reference 2): if an arbitrary load, applied to a body of arbitrary form, grows in proportion to a general parameter λ, then this is sufficient for each element of the body to be under the conditions of simple loading; i.e., for the directed stress hyperboloid to remain fixed at each point of the body.

We emphasize that this is only a sufficient condition. Tests show that for complex loading sufficiently close to a simple loading, the theory based on the following laws gives results close to the true results. Thus if there acts on a body a system of forces of which each in the course of the loading process is a definite constant fraction of the whole, then the loading of each element is simple. If only one force or one uniform pressure p acts, (for example, in a tube, the inward or outward pressure) it may be taken as the parameter λ and simple loading may result from any law of growth with time.

All the different theories of plasticity without exception may be written in the form of a single tensor equation

$$L(D_s) = L'(D_e) \tag{5}$$

where L and L' are linear integro-differential operators related to the deviators D_s and D_e by a parameter λ,

$$\left.\begin{aligned} L(D_s) &= AD_s + B\,\frac{dD_s}{d\lambda} + \ldots + \int_0^{\lambda} CD_s d\lambda + \int_0^{\lambda}\int_0^{\lambda'} DD_s d\lambda' d\lambda + \ldots \\ L'(D_e) &= A'D_e + B'\,\frac{dD_e}{d\lambda} + \ldots + \int_0^{\lambda} C'D_e d\lambda + \int_0^{\lambda}\int_0^{\lambda'} D'D_e d\lambda' d\lambda + \ldots \end{aligned}\right\} \quad (6)$$

in which the coefficients $A, B, C, \ldots A', B', C', \ldots$ are functions of the tensor invariants D_s and D_e and of tensors obtained by the laws of linear transformation, and in addition, the parameters $\lambda, \lambda', \lambda'' \ldots$. It is necessary to add to equation (5) a certain number of scalar relationships between the invariants.

The plasticity theory of St. Venant-Levy-Mises is obtained from equation (5), if all coefficients are zero except A and B, but the ratio $\frac{A'}{B'}$ is determined from the Mises condition; to this one must add a scalar condition: the condition of incompressibility.

The plasticity theory of Prandtl-Reuss is obtained from equation (5) when A, B, and B' are different from zero, and

$$B' = 1$$

$$B = \frac{1}{2G}$$

$$A = \frac{3D_s}{2\sigma_s^2} \cdot \frac{dD_e}{d\lambda}$$

where the dot denotes a scalar product of tensors; to this one must add the condition of incompressibility.

The theory of small elasto-plastic strains, the basis for the development of Hencky and Nadai, is obtained from equation (5) when A and A' are different from zero and the relation between them is established by a known law of strain-hardening $\sigma_i = \Phi(e_i)$ and Hooke's Law $\sigma = 3ke$ for volume strains.

The Handelman-Prager theory presented in the preceding article is obtained from equation (5) if we set

$$B = 1$$

$$B' = 2G$$

$$A = g(\sigma_i)\frac{d\sigma_i}{d\lambda}$$

with the remaining coefficients set equal to zero. In the result we obtain

$$\left.\begin{aligned} 2G\delta e_{xx} &= \delta S_{xx} + g(\sigma_i)\delta\sigma_i S_{xx}, \ . \ . \ . \\ G\delta e_{xy} &= \delta S_{xy} + g(\sigma_i)\delta\sigma_i S_{xy}, \ . \ . \ . \end{aligned}\right\} \tag{7}$$

where σ_i is the stress intensity. Equation (7) contains one undetermined function $g(\sigma_i)$ which must be found from tests. Inasmuch as equation (7) must be true for simple loading for which there must be a known dependency $\sigma_i = \Phi(e_i)$ between the stress intensity σ_i and the strain intensity e_i represented by the tension diagram, it is easy to show the relation

$$g(\sigma_i) = \frac{3G - \frac{d\sigma_i}{de_i}}{\sigma_i \frac{d\sigma_i}{de_i}} \tag{8}$$

Hence the new theory of Prager contains no new experimentally determined functions which would characterize complex loading conditions.

In conclusion we present the following sufficiently obvious theorem which shows the unity of all theories described by equation (5): if the loading of an element of the body is simple and if equation (5) must not show the appearance of relaxation after-effects, creep, or other phenomena connected with time, then it is identical with the equation

$$A(D_s) = A'(D_e) \tag{9}$$

By the condition of the theorem, the directed tensors D_s^* and D_e^* do not depend on the parameter λ, hence

$$\frac{dD_s}{d\lambda} = \frac{d}{d\lambda}\left(\tau_i D_s^*\right) = D_s \frac{d\tau_i}{d\lambda}$$

$$\int_0^\lambda CD_s d\lambda = \int_0^\lambda C\tau_i D_s^* d\lambda = D_s \int_0^\lambda C\tau_i d\lambda$$

Consequently the operator $L(D_s)$ may be written in the form

$$L(D_s) = D_s^* L(\tau_i)$$

The same is also true for $L'(D_e)$, therefore equation (5) takes the form

$$D_s^* L(\tau_i) = D_e^* L'\left(\frac{\gamma_i}{2}\right)$$

But $L(\tau_i)$ and $L'(\gamma_i)$ are invariants and may be denoted by A and A', since the parameter λ evidently must not enter in them, for otherwise a dependency on the time would result; so they must be functions only of τ_i and γ_i.

Thus, all the theories of plasticity represented in equation (5) are identical among themselves for the case of simple loading and equivalent to the simplest of them – the theory of small elasto-plastic strains.

Therefore, to talk about the degrees of exactness of this or that one of them, as do Handelman and Prager, must refer only to the processes of complex loading. In particular, relative to the new theory of Prager previously proposed, one may say that it leads to obvious contradiction with tests since it involves only a single experimental function $g(\sigma_i)$, completely determined only by simple loading tests; functions reflecting the Bauschinger effect, the possibility of rotation of the principal stress axes relative to parts of the body and the strain-hardening effect are not included in the new theory.

There remains only to note that the preceding theory of Prager in which he improves the accuracy of the theory of small elasto-plastic strains

by introduction of a nonlinear tensor equation, is useful in the supplementary calculation of corrections of the second order of smallness.

Translation by E. Z. Stowell
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